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A GENERAL THEORY OF ELASTIC BEAMS

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Abstract—The nonlinear basic equations which govern the motion of beams are developed on the basis of three-dimensional theory of thermo-elastodynamics in terms of a reference state. A general method of expansion together with a variational procedure is used for the formulation. Thus, a hierarchy of one-dimensional approximate theories is consistently established. The geometrical as well as physical nonlinearity are explicitly considered in the analysis. The classical theories of stress and strain in beams are directly deduced from the general results. The theory accommodates the higher order stretching, bending and torsion of non-polar elastic beams of uniform cross section.

1. NOTATION

Throughout the paper, a system of the right-handed Cartesian convected (intrinsic) co-ordinates x_k $(k = 1, 2, 3)$ and Einstein's summation convention are used. Accordingly, repeated Latin indices represent summation over the range (1, 2, 3) and repeated Greek indices are summed over the range (1, 2), unless indices are put within parentheses. An index following a comma stands for partial differentiation with respect to the indicated co-ordinate x_k, while a superposed dot denotes partial differentiation with respect to time t. Also, a star is used to designate prescribed quantities.

Essentially, new quantities are defined when they are first introduced.

The following symbols are used in the text:

d, L	maximum diameter of cross-section and length of beam
$\mathscr{A}$	area of cross-section
$\mathscr{C}$	a Jordan curve which bounds cross-section
x_k	a system of right-handed Cartesian convected co-ordinates
t	time
$\mathbf{u}$	displacement vector
γ_{kl}	Lagrangian strain tensor
ω_{kl}, e_{kl}	rotation and linear strain tensors
$u_k^{(m,n)}, \gamma_{kl}^{(m,n)}$	displacement and strain components of order (m, n)
$\omega_{kl}^{(m,n)}, e_{kl}^{(m,n)}$	rotation and linear strain tensors of order (m, n)
ρ	density of the undeformed body
t_{kl}, s_{kl}	asymmetric Lagrangian and symmetric Kirchhoff stress tensors
$I^{(m,n)}$	moment of inertia of order (m, n)
$T_{kl}^{(m,n)}$	stress resultants of order (m, n)
$\mathbf{f}$	body force vector/unit mass of the undeformed body
$F_k^{(m,n)}, P_k^{(m,n)}, Q_k^{(m,n)}$	body force, external force and effective load of order (m, n), respectively
C_{klmn}	isothermal elastic constants
λ, μ	Lamé's elasticity constants
E, ν	Young's modulus and Poisson's ratio
α_{kl}	strain–temperature constants
$\mathbf{t}$	stress vector measured/unit area of the undeformed body
$\mathbf{n}$	unit outward normal vector to the undeformed position of a surface element in the deformed body, associated with $\mathbf{t}$
$\varepsilon_{\alpha\beta}$	second order permutation symbol
W, Σ	strain energy densities/unit volume and per unit length of the undeformed beam, respectively
$\vartheta, \mathscr{S}$	entire volume and boundary surface of the undeformed beam
$\mathscr{S}_u, \mathscr{S}_\sigma$	surface parts of $\mathscr{S}$, where displacements and tractions are prescribed, respectively

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Θ	increment in temperature
$\mathscr{A}_l, \mathscr{A}_r$	left and right edge surfaces of the undeformed beam
$\mathscr{S}_e, \mathscr{S}_l$	entire edge and lateral surfaces of the undeformed beam
ω	rate of twist
e, e_1	voluminal and aerial dilatations
c	velocity of shear waves in an unbounded medium
c_0	bar velocity
ϕ	warping function
$s_{kl}^{(0)}$	initial stress tensor
dV, dS, dA	elements of volume, lateral surface and face boundary
$d\sigma$	line element along $\mathscr{C}$
$\mathbf{v}$	unit outward drawn vector normal to $\mathscr{C}$

2. INTRODUCTION

IN SPITE of various significant contributions in the literature, a complete nonlinear theory of elastic beams, which provokes further developments and enables one to make plausible assumptions and convenient approximations to meet demands of engineering, is not presently available. An attempt is thus made to develop a consistent fully nonlinear theory of beams within the framework of three-dimensional theory of thermo-elasticity.

The earliest works on beams and columns may be traced back to Bernoulli, and they are referred to Ericksen and Truesdell [1] and Truesdell [2]. These works are chiefly bound to linear elasticity and they are formulated under *ad hoc* simplifying hypotheses. Since the resulting equations are so simple and clear these theories are still being employed notwithstanding the fact that they are lacking in increasing accuracy and estimation of errors. Apparently, some other techniques need to be used to reach a rational theory of beams rather than the method of hypotheses leading to the usual theories. Recently, Gol'denweizer [3] presented a comprehensive survey on the techniques which can be used in a general analysis of structures. In addition, Nigul [4] and Kalinin [5] briefly discussed these techniques as did Green *et al.* [6] and Koiter [7]. Among these techniques, such as the direct method, the asymptotic method and the method of series expansion were exhibited in [6, 8, 9], [4–6, 10–12] and [13–17], respectively. Along this line, one should also mention Ref. [18] for a general treatment of the approximate methods of analysis in elasticity theory, involving existence and uniqueness theorems.

In the present analysis, the method of series expansion is used in the form given by Mindlin [14] who recapitulated the method from the works of Cauchy [19] and Poisson [20]. Mindlin and his co-authors [13–15, 21, 22] and the present author [16, 17] extensively used the method in the formulation of one- and two-dimensional continuum theories. The method involves the series expansion of all field quantities in terms of the appropriate co-ordinates. The series expansion converts three-dimensional field equations of elasticity into a hierarchy of one-dimensional approximate equations with the aid of either the variational method of Kirchhoff [23] or a direct method of integration. Thus, the governing equations are consistently obtained. The application of the method is accomplished in a tractable and straightforward manner. Without attempting to be exhaustive, the most pertinent references to the present paper are mentioned here as follows.

By the use of a power series representation for stresses in terms of a small thickness parameter, Hay [24] was the first to formulate consistently a finite displacement–small strain theory of elastic rods. Mindlin [15] derived a linear theory of isotropic elastic beams by means of the variational procedure of [23], with which he examined the uniqueness of solutions in Neumann's sense. Mindlin and McNiven [22] studied the axially symmetric motions of an elastic rod of circular cross-section expressing displacement components in series of Jacobi polynomials in terms of the radial co-ordinate. By the use of a series of Legendre polynomials for displacement components in terms of lateral co-ordinates, a one-dimensional theory of elastic bars of rectangular cross-section has been obtained by Medick [25, 26] and Hertelendy [27] who presented some experimental results as well. As a generalized plane stress problem, the elastic beam theory has been studied by Soler [28] and Hashin [29]. A series of Legendre polynomials is used for isotropic rectangular strips in [28], while a power series is used for plane anisotropic rectangular beams in [29]. The authors of [25–27] were primarily concerned with the dynamic behavior of bars and those of [28, 29] studied the static behavior. In addition to the references above, other references utilizing linear theory include Warner [30] and more recently Bleustein and Stanley [31] as well as Green *et al.* [6]. The works of the latter three authors [8] and of Antman and Warner [32] should be mentioned among the recent contributions to the nonlinear theory. Starting with a series representation for the position vector, an isothermal theory of rods is formulated in [32], while a thermo-dynamical theory is given in [8]. Mention may also be made of the exact theories of Ericksen and Truesdell [1] and Green [33]. However, these theories did not include the constitutive relations.

This paper aims at a rigorous derivation of the beam equations from the three-dimensional field equations of elasticity, including large displacements and large angles of rotations. A generalized variational procedure deduced from the Hamiltonian principle and a method of series expansion for kinematic variables are used in the formulation. The effects of inertia, both transverse and in-plane, and of temperature are included as is the influence of heterogeneity and anisotropy of the material (thus making it applicable, for example, to composite materials). The theory accommodates the higher order stretching, bending and torsion of elastic beams of uniform cross-section. The governing equations consist of the macroscopic equations of motion, the natural boundary conditions, the strain–displacement equations and the constitutive relations.

The kinematic variables are presented in the next section. Section 4 deals with the strain–displacement relations. The variational procedure is exhibited in Section 5. The load and stress resultants, and the constitutive relations are given in Sections 6 and 7, respectively. The boundary conditions and the macroscopic equations of motion for non-polar beams of uniform cross-section are extensively studied in Section 8. A linear theory of beams and its simplified versions are presented in Section 9. The last section is devoted to concluding remarks.

3. KINEMATIC VARIABLES

An initially slender beam of constant cross-section is treated in this analysis. The beam is referred to a system of right-handed Cartesian convected (intrinsic) co-ordinates x_k. The axes x_1 and x_2 are chosen as the principal axes of the cross-section. The locus of the centroids of cross-sections is a straight line in the undeformed beam, and it is taken as the axis x_3. The cross-section of the beam is bounded by a simply-connected Jordan curve $\mathscr{C}$, i.e. sufficiently smooth and non-intersecting. In this context, a cylindrical beam with no singularities of any type is supposed to be present. Consequently, the displacement and stress fields are continuous throughout the beam space ϑ. Moreover, the use of the inequality:

$$\frac{d}{L} \ll 1 \tag{3.1}$$

where d and L are respectively the maximum diameter of cross-section and the length of beam, allows the beam to be treated as a one-dimensional mathematical model of a three-dimensional body.

The displacement components of a generic point in ϑ, based on the above considerations, can be represented by

$$u_k(\mathbf{x}, t) = \sum_{m+n=0}^{\infty} P_m^{(k)}(x_1) \,.\, Q_n^{(k)}(x_2) \,.\, u_k^{(m,n)}(x_3, t) \tag{3.2a}$$

with

$$P_0^{(k)}(x_1) = Q_0^{(k)}(x_2) = 1. \tag{3.2b}$$

Here, from the mathematical standpoint, a separation of variables solution is sought for the nonlinear field equations which are presented in the next sections. Therefore, the vector functions in (3.2) are unknown *a priori* and independent functions defined in ϑ. Also, it is assumed that $u_k^{(m,n)}$ exists and is a function of class C^2 (C^n represents the functions with derivatives of order n, with respect to space co-ordinates x_k and time t). In the subsequent analysis, the two functions of the form:

$$P_m^{(k)}(x_1) = x_1^m, \qquad Q_n^{(k)}(x_2) = x_2^n \tag{3.2c}$$

are to be used. If the displacement vector $\mathbf{u}$ is analytic with respect to the aerial co-ordinates x_α in ϑ, (3.2) can be regarded as a Taylor expansion of $\mathbf{u}$, which is uniformly convergent in the closed region ϑ. However in this case, $\mathbf{u}^{(m,n)}$ is an independent function. In (3.2), P_m and Q_n, for instance, could be Legendre polynomials, Jacobi polynomials and/or any other appropriate functions.

In contrast to the customary beam theories, the Bernoulli–Euler hypothesis is abrogated here by virtue of (3.2), i.e. sections which are plane and perpendicular to the centroid locus in the undeformed beam do not necessarily remain so in the deformed beam and suffer no strains in their planes (see, e.g. Boley and Weiner [34]).

4. STRAIN–DISPLACEMENT RELATIONS

The Lagrangian strain tensor γ_{kl} is expressed in terms of the displacement components [35]:

$$\gamma_{kl} = \tfrac{1}{2}(u_{k,l} + u_{l,k} + u_{r,k}u_{r,l}) \tag{4.1}$$

and

$$\gamma_{kl} = e_{kl} + \tfrac{1}{2}(e_{rk} + \omega_{rk})(e_{rl} + \omega_{rl}). \tag{4.2}$$

Here, the linear strain tensor e_{kl} and the rotation tensor ω_{kl} are given by

$$e_{kl} = e_{lk} = \tfrac{1}{2}(u_{k,l} + u_{l,k}), \qquad \omega_{kl} = -\omega_{lk} = \tfrac{1}{2}(u_{k,l} - u_{l,k}). \tag{4.3}$$

The series expansion in all displacement components as in (3.2) and the relations given above imply a strain distribution of the form:

$$\gamma_{kl}(\mathbf{x}, t) = \sum_{m+n=0}^{\infty} x_1^m x_2^n \gamma_{kl}^{(m,n)}(x_3, t) \tag{4.4a}$$

where

$$\gamma_{kl}^{(m,n)} = e_{kl}^{(m,n)} + \tfrac{1}{2} \sum_{p+q=0}^{m+n} [e_{rk}^{(m-p,n-q)} + \omega_{rk}^{(m-p,n-q)}][e_{rl}^{(p,q)} + \omega_{rl}^{(p,q)}] \tag{4.4b}$$

with

$$\begin{aligned}
e_{\alpha\beta}^{(m,n)} &= \tfrac{1}{2}[(m+1)(\delta_{1\alpha}u_\beta^{(m+1,n)} + \delta_{1\beta}u_\alpha^{(m+1,n)}) + (n+1)(\delta_{2\alpha}u_\beta^{(m,n+1)}) + \delta_{2\beta}u_\alpha^{(m,n+1)})] \\
e_{\alpha 3}^{(m,n)} &= \tfrac{1}{2}[u_{\alpha,3}^{(m,n)} + (m+1)\delta_{1\alpha}u_3^{(m+1,n)} + (n+1)\delta_{2\alpha}u_3^{(m,n+1)}] \\
e_{33}^{(m,n)} &= u_{3,3}^{(m,n)}, \qquad \omega_{33}^{(m,n)} = 0 \\
\omega_{\alpha\beta}^{(m,n)} &= \tfrac{1}{2}[(m+1)(\delta_{1\beta}u_\alpha^{(m+1,n)} - \delta_{1\alpha}u_\beta^{(m+1,n)}) + (n+1)(\delta_{2\beta}u_\alpha^{(m,n+1)} - \delta_{2\alpha}u_\beta^{(m,n+1)})] \\
\omega_{\alpha 3}^{(m,n)} &= \tfrac{1}{2}[u_{\alpha,3}^{(m,n)} - (m+1)\delta_{1\alpha}u_3^{(m+1,n)} - (n+1)\,\delta_{2\alpha}u_3^{(m,n+1)}].
\end{aligned} \tag{4.4c}$$

Here, $\gamma_{kl}^{(m,n)}$, $u_k^{(m,n)}$, $e_{kl}^{(m,n)}$ and $\omega_{kl}^{(m,n)}$ are henceforth termed the Lagrangian strain tensor, the displacement vector, the linear strain tensor and the rotation tensor of order (m, n), respectively.

The linear version of (4.1) is simply expressed as

$$\gamma_{kl} = e_{kl} \tag{4.5a}$$

with

$$\gamma_{kl}^{(m,n)} = e_{kl}^{(m,n)} \tag{4.5b}$$

in (4.4).

5. VARIATIONAL PROCEDURE

When the motion of the non-polar continuum is referred to a fixed system of Cartesian axes, the equations of local balance of momentum given in [36] are:

$$t_{kl,k} + \rho(f_l - a_l) = 0 \tag{5.1}$$

with

$$t_{kl} = s_{kr}(\delta_{lr} + u_{l,r}). \tag{5.2}$$

Here, a_k and f_k, respectively, denote the Lagrangian components of the acceleration and the body force measured per unit mass of the undeformed body. ρ indicates the density of the undeformed body. t_{kl} and s_{kl} represent the asymmetric Lagrangian and symmetric Kirchhoff stress tensors measured per unit area of the undeformed body, respectively. When the stress vector $\mathbf{t}$/unit area of the undeformed body, associated with a surface in the deformed body, is referred to the base vectors in the deformed body, s_{kl} arises, while if $\mathbf{t}$ is referred to the base vectors in the undeformed body, t_{kl} ensues.

Let $\mathbf{t}^*$ and $\mathbf{u}^*$ be the prescribed values of the stress and displacement vectors on the boundary surface. Thus, the boundary conditions can be written in the form:

$$u_k^* - u_k = 0 \quad \text{on } \mathscr{S}_u \tag{5.3}$$

and

$$t_k^* - t_k = 0 \quad \text{on } \mathscr{S}_\sigma \tag{5.4}$$

with

$$t_k = t_{lk} n_l. \tag{5.5}$$

Here, t_k is the components of $\mathbf{t}$, associated with a surface in the deformed body, the unit outward drawn normal of which is n_k in its undeformed position. $\mathscr{S}_u$ and $\mathscr{S}_\sigma$ are the parts of the entire boundary surface $\mathscr{S}$, where the displacement and stress vectors are prescribed, respectively.

Let t_0 and t_1 be two arbitrary instants of time and δ indicate the variation. Then, following Love [37], there can be deduced the equation:

$$\delta J = \int_{t_0}^{t_1} \mathrm{d}t \left\{ \int_{\vartheta} [t_{kl,k} + \rho(f_l - a_l)]\, \delta u_l \, \mathrm{d}v + \int_{\mathscr{S}_u} (u_k - u_k^*)\, \delta t_k \, \mathrm{d}S + \int_{\mathscr{S}_\sigma} (t_k^* - t_k)\, \delta u_k \, \mathrm{d}S \right\} = 0 \tag{5.6}$$

from the usual version of the Hamiltonian principle. Since the variations δu_k and δt_k are quite arbitrary, the coefficients of these variations under the integral sign must vanish separately over $\mathscr{S}$ and at all points in the interior of ϑ. This leads to (5.1)–(5.4) and thus verifies the variational formulation.

The variational integral (5.6) leads to the macroscopic equations of motion and to the natural boundary conditions of beam for the case of large displacements and large angles of rotation. This will be shown in the following sections.

6. LOAD AND STRESS RESULTANTS

The following terminology is used in the subsequent analysis. Hence, it seems appropriate to define them beforehand.

Let us define a body force resultant of order (m, n):

$$F_k^{(m,n)} = \int_{\mathscr{A}} \rho f_k x_1^m x_2^n \, \mathrm{d}A \tag{6.1}$$

a stress resultant of order (m, n):

$$T_{kl}^{(m,n)} = \int_{\mathscr{A}} x_1^m x_2^n s_{kl} \, \mathrm{d}A \tag{6.2}$$

surface loads of order (m, n):

$$P_k^{(m,n)} = \oint_{\mathscr{C}} x_1^m x_2^n v_\alpha s_{\alpha k} \, \mathrm{d}s \tag{6.3a}$$

$$R_k^{(m,n)} = \sum_{p+q=0}^{\infty} [(p \,.\, P_1^{(m+p-1,n+q)} + q \,.\, P_2^{(m+p,n+q-1)}) u_k^{(p,q)} + P_3^{(m+p,n+q)} u_{k,3}^{(p,q)}] \tag{6.3b}$$

and an effective load of order (m, n):

$$Q_k^{(m,n)} = F_k^{(m,n)} + P_k^{(m,n)} + R_k^{(m,n)}. \tag{6.4}$$

In these relationships, $\mathscr{A}$ is the area of cross-section, $\mathrm{d}s$ is the line element of $\mathscr{C}$ and $\nu_\alpha = \varepsilon_{\alpha\beta}(\mathrm{d}x_\beta/\mathrm{d}s)$ is the unit exterior normal vector on $\mathscr{C}$.

Similarly, an acceleration resultant of order (m, n):

$$U_k^{(m,n)} = \sum_{p+q=0}^{\infty} I^{(m+p,\,n+q)} u_k^{(p,q)} \tag{6.5}$$

prescribed stress resultants of order (m, n):

$$T_k^{*(m,n)} = \int_{\mathscr{A}} x_1^m x_2^n t_k^* \,\mathrm{d}A, \qquad P_k^{*(m,n)} = \oint_{\mathscr{C}} x_1^m x_2^n t_k^* \,\mathrm{d}s \tag{6.6}$$

and an aerial moment of inertia of order (m, n):

$$I^{(m,n)} = \int_{\mathscr{A}} x_1^m x_2^n \,\mathrm{d}A \tag{6.7}$$

are defined. Here, it is pertinent to note that (6.7) yields the usual quantities of elementary beam theory as

$$\begin{gathered} I^{(0,0)} = A \\ I^{(1,0)} = I^{(0,1)} = I^{(1,1)} = 0, \qquad I^{(2,0)} = I_1, \qquad I^{(0,2)} = I_2. \end{gathered} \tag{6.8}$$

Since the principal axes x_α were situated at the centroid of cross-section $I^{(1,0)}$, $I^{(0,1)}$ and $I^{(1,1)}$ readily vanish. Moreover, the following relations hold for a symmetric cross-section with respect to x_1:

$$I^{(m,n)} = \delta_{nn_0} I_{(m,n)} \tag{6.9a}$$

with respect to x_2:

$$I^{(m,n)} = \delta_{mm_0} I_{(m,n)} \tag{6.9b}$$

and with respect to x_1 and x_2:

$$I^{(m,n)} = \delta_{mm_0} \cdot \delta_{nn_0} \cdot I_{(m,n)} \tag{6.9c}$$

where m_0 and n_0 stand for any even integer and δ_{mn} is the usual Kronecker delta.

7. CONSTITUTIVE EQUATIONS

In the case of a perfectly elastic body, a strain energy function or elastic potential W does exist [36], measured per unit volume of the undeformed body and it yields:

$$s_{kl} = \frac{1}{2}\left(\frac{\partial W}{\partial \gamma_{kl}} + \frac{\partial W}{\partial \gamma_{lk}}\right). \tag{7.1}$$

This constitutive relation may also be used for a Hencky type elasto-plastic body as remarked in [35].

By the use of (4.4), (6.2) and (7.1), the constitutive relations for the stress resultants are obtained in the form:

$$T_{kl}^{(m,n)} = \frac{1}{2}\left(\frac{\partial \sum}{\partial \gamma_{kl}^{(m,n)}} + \frac{\partial \sum}{\partial \gamma_{lk}^{(m,n)}}\right) \tag{7.2}$$

where

$$\sum = \int_{\mathscr{A}} W \,\mathrm{d}A. \tag{7.3}$$

Here, $\sum$ denotes a strain energy function measured per unit length of the undeformed beam. The fully nonlinear constitutive relations are expressed by (7.1)–(7.3).

The generalized linear constitutive relations given in [38] are:

$$s_{kl} = C_{klmn}(\gamma_{mn} - \alpha_{mn}\Theta) \tag{7.4}$$

for an initially perfect elastic, anisotropic and heterogeneous beam material subjected to a prescribed steady temperature field $\Theta(x_k)$. The corresponding elastic potential can be written as:

$$W = \tfrac{1}{2}C_{klmn}(\gamma_{kl} - \alpha_{kl}\Theta)(\gamma_{mn} - \alpha_{mn}\Theta). \tag{7.5}$$

Here, C_{klmn} and α_{kl} are the isothermal elastic constants and the thermal expansion coefficients at constant stress, respectively. From energy considerations it follows that

$$C_{klmn} = C_{mnkl} = C_{lkmn}, \qquad \alpha_{kl} = \alpha_{lk}. \tag{7.6}$$

In the case of isotropic material, they reduce to:

$$\begin{aligned} \alpha_{kl} &= \alpha\delta_{kl} \\ C_{klmn} &= \lambda\delta_{kl}\delta_{mn} + \mu(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}) \end{aligned} \tag{7.7}$$

where α is the coefficient of linear thermal expansion, λ and μ are Lamé's constants. These in turn, can be expressed in terms of Young's modulus E and Poisson's ratio ν as

$$\lambda = \frac{2\mu\nu}{(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)} \tag{7.8a}$$

with

$$\mu > 0, \qquad 3\lambda + 2\mu > 0. \tag{7.8b}$$

With the aid of (4.4), (6.2) and (7.4), the linear macroscopic constitutive equations are obtained:

$$T_{kl}^{(m,n)} = C_{klrs} \sum_{p+q=0}^{\infty} I^{(m+p,\,n+q)}(\gamma_{rs}^{(p,q)} - \alpha_{rs}\Theta^{(p,q)}) \tag{7.9}$$

and

$$T_{kl}^{(m,n)} = \sum_{p+q=0}^{\infty} I^{(m+p,\,n+q)}[\lambda(\gamma_{rr}^{(p,q)} - 3\alpha\Theta^{(p,q)})\delta_{kl} + 2\mu(\gamma_{kl}^{(p,q)} - \alpha\Theta^{(p,q)}\delta_{kl})] \tag{7.10}$$

for anisotropic and isotropic beam materials, respectively. Here, the temperature field is taken in the form

$$\Theta(x_k) = \sum_{m+n=0}^{\infty} x_1^m x_2^n \Theta^{(m,n)}(x_3). \tag{7.11}$$

Finally, the strain energy:

$$\sum = \int_{\mathscr{A}} W \, dA = \tfrac{1}{2} C_{klmn} \sum_{p+q=0}^{\infty} \sum_{r+s=0}^{\infty} I^{(p+q,r+s)}(\gamma_{kl}^{(p,q)} - \alpha_{kl}\Theta^{(p,q)})(\gamma_{mn}^{(r,s)} - \alpha_{mn}\Theta^{(r,s)}) \tag{7.12}$$

and the kinetic energy:

$$K = \int_{\mathscr{A}} \left(\frac{1}{2} \rho a_k a_k \right) dA = \frac{1}{2} \rho \sum_{p+q=0}^{\infty} \sum_{r+s=0}^{\infty} I^{(p+r,q+s)} \ddot{u}_k^{(p,q)} \ddot{u}_k^{(r,s)} \tag{7.13}$$

are obtained per unit length of the undeformed beam.

8. BEAM EQUATIONS OF MOTION

We now proceed to develop the nonlinear equations of motion of beams together with the natural boundary conditions in terms of the displacement field defined previously in Section 3 and the effective load, stress, body force and acceleration resultants given in Section 6. For this purpose, the variational equation (5.6) is evaluated.

Consider first the volume integral in (5.6), namely

$$\delta J_1 = \int_{t_0}^{t_1} dt \int_0^L dx_3 \int_{\mathscr{A}} [t_{kl,k} + \rho(f_l - a_l)] \, \delta u_l \, dA. \tag{8.1}$$

Substituting the series expansion (3.2) into this integral, performing the integrations over a cross-section of the beam and replacing the stress and load resultants (6.1)–(6.8), one obtains:

$$\delta J_1 = \int_{t_0}^{t_1} dt \int_0^L dx_3 \sum_{m+n=0}^{\infty} (T_{3l,3}^{(m,nn)} - m \,.\, T_{1l}^{(m-1,n)} - n \,.\, T_{2l}^{(m,n-1)} + N_l^{(m,n)}$$
$$+ Q_l^{(m,n)} - \rho \ddot{U}_l^{(m,n)}) \, \delta u_l^{(m,n)} \tag{8.2}$$

where

$$N_k^{(m,n)} = \sum_{p+q=0}^{\infty} \{[mpT_{11}^{(m+p-2,n+q)} + (np+mq)T_{12}^{(m+p-1,n+q-1)} + qnT_{22}^{(m+p,n+q-1)}$$
$$+ pT_{31,3}^{(m+p-1,n+q)} + qT_{32,3}^{(m+p,n+q-1)}]u_k^{(p,q)} + T_{33}^{(m+p,n+q)}u_{k,33}^{(p,q)}$$
$$+ [(p+m)T_{13}^{(m+p-1,n+q)} + (q+n)T_{23}^{(m+p,n+q-1)} + T_{33,3}^{(m+p,n+q-1)}]u_{k,3}^{(p,q)}\}. \tag{8.3}$$

The surface integrals in (5.6) are:

$$\partial J_2 = \int_{t_0}^{t_1} dt \int_{\mathscr{S}_u} (u_k - u_k^*) \, \delta t_k \, dS \tag{8.4}$$

and

$$\delta J_3 = \int_{t_0}^{t_1} \mathrm{d}t \left\{ \int_{\mathscr{S}_e} (t_k^* - t_k)\, \delta u_k \, \mathrm{d}A + \int_{\mathscr{S}_t} (t_k^* - t_k)\, \delta u_k \, \mathrm{d}S \right\}. \tag{8.5}$$

Here, the surface part $\mathscr{S}_u$, where the displacement vector is prescribed, is taken as a portion of the lateral surface $\mathscr{S}_l$, while $\mathscr{S}_\sigma$, where the stress vector is prescribed, is taken as the remaining portion of $\mathscr{S}_l$ and both the left face boundary $\mathscr{A}_l$ and the right face boundary $\mathscr{A}_r$. Thus, one reads

$$\mathscr{S} = \mathscr{S}_u \cup \mathscr{S}_\sigma, \quad \mathscr{S}_u \cap \mathscr{S}_\sigma = 0, \quad \mathscr{S}_u \cup \mathscr{S}_t = \mathscr{S}_l, \quad \mathscr{S}_\sigma = \mathscr{S}_t \cup \mathscr{S}_e, \quad \mathscr{S}_e = \mathscr{A}_r \cup \mathscr{A}_l. \tag{8.6}$$

Carrying out the integrations of (8.4) and (8.5) as in the volume integral, the equations:

$$\delta J_2 = \int_{t_0}^{t_1} \mathrm{d}t \int_{\mathscr{S}_u} \sum_{m+n=0}^{\infty} x_1^m x_2^n (u_k^{(m,n)} - u_k^{*(m,n)})\, \delta t_k \, \mathrm{d}S \tag{8.7}$$

and

$$\delta J_3 = \int_{t_0}^{t_1} \mathrm{d}t \left\{ \sum_{m+n=0}^{\infty} [T_k^{*(m,n)} - n_3 (T_{3k}^{(m,n)} + N_{3k}^{(m,n)})]\, \delta u_k^{(m,n)} \right.$$

$$\left. + \int_0^L \mathrm{d}x_3 \sum_{m+n=0}^{\infty} [P_k^{*(m,n)} - (P_k^{(m,n)} + R_k^{(m,n)})]\, \delta u_k^{(m,n)} \right\} \tag{8.8a}$$

are obtained. In (8.8a), $N_{3k}^{(m,n)}$ is defined to be

$$N_{3k}^{(m,n)} = \sum_{p+q=0}^{\infty} [(pT_{31}^{(m+p-1,n+q)} + qT_{32}^{(m+p,n+q-1)}) u_k^{(p,q)} + T_{33}^{(m+p,n+q)} u_{k,3}^{(p,q)}]. \tag{8.8b}$$

Setting the variational integrals (8.2), (8.7) and (8.8) equal to zero for the arbitrary and independent variations of the displacement and traction components, the hierarchy of the one-dimensional approximate equations of motion and the corresponding natural boundary conditions are found and given as follows.

$$\begin{aligned} T_{3k,3}^{(m,n)} - m\,.\,T_{1k}^{(m-1,n)} - n\,.\,T_{2k}^{(m,n-1)} + N_k^{(m,n)} + Q_k^{(m,n)} - \rho \ddot{U}_k^{(m,n)} &= 0 \\ u_k^{(m,n)} - u_k^{*(m,n)} &= 0 \quad \text{on } \mathscr{S}_u \\ P_k^{*(m,n)} - (P_k^{(m,n)} + R_k^{(m,n)}) &= 0 \quad \text{on } \mathscr{S}_t \\ T_k^{*(m,n)} \pm (T_{3k}^{(m,n)} + N_{3k}^{(m,n)}) &= 0 \quad \text{on } \begin{Bmatrix} \mathscr{A}_l \\ \mathscr{A}_r \end{Bmatrix}. \end{aligned} \tag{8.9}$$

These equations are henceforth called the macroscopic equations of motion and the natural boundary conditions of order (m, n).

Thus far, a fully nonlinear theory of beams has been established. This consists of the strain–displacement relations (4.4), the constitutive equations (7.2) and the equations of motion and the natural boundary conditions (8.9). In addition, it should be noted that the initial conditions for the displacement components, i.e. $u_k^{(m,n)}$ and $\dot{u}_k^{(m,n)}$ must be prescribed at $t = t_0$, as is customary in the use of the Hamiltonian principle. At this point, there exists an infinite number of equations (4.4), (7.2) and (8.9) for an infinite number of unknowns

$(u_k^{(m,n)}, \gamma_{kl}^{(m,n)}, T_{kl}^{(m,n)})$. Thus, the governing equations are not tractable and formally determinate. In order to obtain a deterministic hierarchy of the governing equations, these infinite number of equations with their infinite number of unknowns must be consistently reduced to a finite number of equations and unknowns by the truncation of the infinite series.

From the foregoing analysis, it is evident that the theory, in essence, is based on the series expansion (3.2) whose the terms $u_k^{(m,n)}$ are already assumed to exist in Section 3. Thus, the theory of order (M, N) is defined by either

$$u_k = \sum_{m=0}^{M} \sum_{n=0}^{N} x_1^m x_2^n u_k^{(m,n)} \tag{8.10a}$$

or (3.2) together with the condition:

$$u_k^{(m,n)} = 0 \quad \text{for all} \quad m \geq M+1, \qquad n \geq N+1. \tag{8.10b}$$

Accordingly, only those quantities involved in (8.10) are considered in the governing equations. In view of (8.10), the number of unknown displacement components $u_k^{(m,n)}$ is now reduced to $3\mathcal{N}$, $[\mathcal{N} = (M+1)(N+1)]$. The required equations needed to determine these unknowns are consistently obtained by means of the variational equation (5.6).

Besides the one described above, mention should be made of another deterministic theory which is simply defined by the condition:

$$u_k = \sum_{m+n=0}^{\mathcal{N}} x_1^m x_2^n u_k^{(m,n)} \tag{8.11a}$$

and

$$u_k^{(m,n)} = 0 \quad \text{for all} \quad (m+n) \geq \mathcal{N}+1. \tag{8.11b}$$

A similar definition is introduced for the higher order theories of plates [14] as well as those of rods [32]. Nevertheless, this can not be a pertinent definition for beams, since it tacitly assumes that both of the lateral co-ordinates have the same weight in the series expansion (3.2). For a rectangular strip or a thin beam, (3.2) degenerates into a series of the form:

$$u_k = \sum_{n=0}^{N} u_k^{(n)}(x_3, t) P_n^{(k)}(x_1) \tag{8.12}$$

as has successfully been used in [28, 30]. The expansion (8.12) is not obtainable from (8.11) as a special case, whereas (8.10) clearly contains both cases.

The governing equations of order (M, N) are coupled and nonlinear partial differential equations, and they must be simultaneously examined for each order (M, N). These equations become ordinary differential equations when static equilibrium is considered, i.e. time is dropped out as an independent variable.

9. A LINEAR BEAM THEORY

A general theory which characterizes the nonlinear behavior of elastic beams has been formulated in the preceding sections. The linearized versions of the theory are now presented.

Dropping out all nonlinear terms in (8.9), namely

$$N_k^{(m,n)} = N_{3k}^{(m,n)} = R_k^{(m,n)} = 0 \tag{9.1}$$

the equations of motion and the natural boundary conditions reduce to

$$\begin{aligned} T_{3k,3}^{(m,n)} - m\,.\,T_{1k}^{(m-1,n)} - n\,.\,T_{2k}^{(m,n-1)} + F_k^{(m,n)} + P_k^{(m,n)} - \rho \ddot{U}_k^{(m,n)} &= 0 \\ u_k^{(m,n)} - u_k^{*(m,n)} &= 0 \quad \text{on } \mathscr{S}_u \\ P_k^{*(m,n)} - P_k^{(m,n)} &= 0 \quad \text{on } \mathscr{S}_t \\ T_k^{*(m,n)} \pm T_{3k}^{(m,n)} &= 0 \quad \text{on } \begin{Bmatrix} \mathscr{A}_l \\ \mathscr{A}_r \end{Bmatrix}. \end{aligned} \tag{9.2}$$

These equations together with (4.5) constitute a beam theory which is geometrically linear but physically nonlinear as far as the constitutive equations (7.2) are concerned. Moreover, a fully linear theory of elastic beams can be deduced by replacing the nonlinear constitutive equations (7.2) by either (7.9) for the case of anisotropy or (7.10) for that of isotropy. This can be considered as a generalized version of the Timoshenko beam theory and it includes all of the effects of shear and rotatory inertia. By means of further reduction; i.e. by dropping out the thermal effects in the linear field equations, the Mindlin [15] theory of elastic and isotropic beams can be derived.

Similarly, with the aid of (9.2) other well-known beam theories can readily be obtained as is exhibited in the remainder of this article.

9.1 *Bernoulli beam theory*

This familiar theory is particularly applicable to longitudinal vibrations in beams. In the present notation, it corresponds to a linear theory of isotropic beams of order (0, 0).

Setting all stress, strains and body force components equal to zero except for $T_{33}^{(0,0)}$ and $\gamma_{11}^{(0,0)}$, $\gamma_{22}^{(0,0)}$, $\gamma_{33}^{(0,0)}$, one then obtains the strain–displacement relations:

$$\gamma_{11}^{(0,0)} = u_1^{(1,0)}, \gamma_{22}^{(0,0)} = u_2^{(0,1)}, \gamma_{33}^{(0,0)} = u_{3,3}^{(0,0)} = W_{,3} = W_{,z} \tag{9.3}$$

the constitutive equations:

$$\begin{aligned} T_{(\alpha\alpha)}^{(0,0)} &= A(\lambda\gamma_{kk}^{(0,0)} + 2\mu\gamma_{(\alpha\alpha)}^{(0,0)}) = 0 \\ T_{33}^{(0,0)} &= A(\lambda\gamma_{kk}^{(0,0)} + 2\mu\gamma_{33}^{(0,0)}) = N \end{aligned} \tag{9.4}$$

and the stress equation of motion:

$$\frac{\partial N}{\partial z} + P - \rho A \frac{\partial^2 W}{\partial t^2} = 0 \tag{9.5a}$$

where

$$P = P_3^{(0,0)}. \tag{9.5b}$$

Here, (4.5), (7.10) and (9.2) are used. By solving (9.4) for the aerial dilatation e_1, one easily computes the voluminal dilatation e as

$$e_1 = \gamma_{\alpha\alpha}^{(0,0)} = -\frac{\lambda}{\lambda+\mu} \cdot \gamma_{33}^{(0,0)}, \qquad e = \gamma_{kk}^{(0,0)} = \frac{\mu}{\lambda+\mu}\gamma_{33}^{(0,0)}$$

and

$$N = A \cdot \frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)} \cdot \gamma_{33}^{(0,0)}. \tag{9.6}$$

Substituting (7.8) and (9.3) into (9.6), one finally arrives at the macroscopic constitutive equation:

$$N = AE\frac{\partial W}{\partial z} \tag{9.7}$$

and the displacement equation of motion:

$$\frac{\partial^2 W}{\partial z^2}+\frac{P}{AE}-\bar{c}^2 \cdot \frac{\partial^2 W}{\partial t^2} = 0 \tag{9.8a}$$

with

$$c^2 = \frac{E}{\rho} \tag{9.8b}$$

which denotes the bar velocity.

9.2 *Timoshenko beam theory*

Equation (9.2) is now used in the formulation of a linear theory of order (1, 0). For this particular case, the cross-section and the loading are assumed to be symmetric about the principal plane (x_1–x_3). Setting all stress, strains and body force components equal to zero except for $T_{13}^{(0,0)}$, $T_{33}^{(1,0)}$ and $\gamma_{13}^{(0,0)}$, $\gamma_{11}^{(1,0)}$, $\gamma_{22}^{(1,0)}$, $\gamma_{33}^{(1,0)}$, one can then write the strain components as:

$$\begin{aligned} &\gamma_{13}^{(0,0)} = \frac{1}{2}\left(\frac{\partial \eta}{\partial z}-\psi\right) \\ &\gamma_{11}^{(1,0)} = 2u_1^{(2,0)}, \qquad \gamma_{22}^{(1,0)} = u_2^{(1,1)}, \qquad \gamma_{33}^{(1,0)} = -\psi_{,z} \end{aligned} \tag{9.9a}$$

where

$$\eta = u_1^{(0,0)}, \qquad \psi = -u_3^{(1,0)} \tag{9.9b}$$

the constitutive relations:

$$\begin{aligned} T_{13}^{(0,0)} &= Q = 2A\mu\gamma_{13}^{(0,0)}, \qquad T_{23}^{(0,0)} = T_{12}^{(0,0)} = 0 \\ T_{(\alpha\alpha)}^{(1,0)} &= I_1[\lambda\gamma_{kk}^{(1,0)}+2\mu\gamma_{(\alpha\alpha)}^{(1,0)}] = 0 \\ T_{33}^{(1,0)} &= I_1[\lambda\gamma_{kk}^{(1,0)}+2\mu\gamma_{33}^{(1,0)}] = M \end{aligned} \tag{9.10}$$

and the loads:

$$P_1^{(0,0)} = R, \qquad P_3^{(1,0)} = T. \tag{9.11}$$

In a manner similar to the Bernoulli beam theory, the aerial dilatation is calculated from (9.10) and has the form:

$$e_1 = \gamma_{\alpha\alpha}^{(1,0)} = -\frac{\lambda}{\lambda+\mu} \cdot \gamma_{33}^{(1,0)}. \tag{9.12}$$

Thus, the first equation of (9.10) can be written as:

$$Q = \mu A\left(\frac{\partial \eta}{\partial z} - \psi\right), \qquad M = -EI_1\psi_{,z}. \tag{9.13}$$

The equations of motion (9.2) now reduce to

$$\begin{aligned} &\frac{\partial Q}{\partial z} + R - \rho A\frac{\partial^2 \eta}{\partial t^2} = 0 \\ &\frac{\partial M}{\partial z} - Q + T + \rho I_1\frac{\partial^2 \psi}{\partial t^2} = 0. \end{aligned} \tag{9.14}$$

With the aid of (9.9) and (9.13), the displacement equations of motion can be expressed, (cf. [39]), as

$$\begin{aligned} \frac{\partial^2 \psi}{\partial z^2} - \varkappa^2\left(\psi - \frac{\partial \eta}{\partial z}\right) - \frac{T}{EI_1} - \bar{c}^2 \cdot \frac{\partial^2 \psi}{\partial t^2} = 0 \\ \frac{\partial}{\partial z}\left(\frac{\partial \eta}{\partial z} - \psi\right) + \frac{R}{\mu A} - \bar{c}_0^2\frac{\partial^2 \eta}{\partial t^2} = 0 \end{aligned} \tag{9.15a}$$

with

$$\varkappa^2 = \frac{A}{I_1} \cdot \frac{\lambda + \mu}{(3\lambda + 2\mu)} = \frac{A}{2I_1(1+\nu)}, \qquad c^2 = 2(1+\nu)c_0^2. \tag{9.15b}$$

Here, c_0 is the velocity of shear waves in an unbounded medium.

It is appropriate to note that the usual correction factor k^2 appears in the coefficient of Q, e.g. for (9.12):

$$Q = k^2\mu A\left(\frac{\partial \eta}{\partial z} - \psi\right) \tag{9.16}$$

and also in (9.14). This factor is studied in detail by Mindlin and Deresiewicz [40].

10. DISCUSSION

A rigorous derivation of the dynamical theory of beams has been obtained within the framework of the three-dimensional nonlinear theory of thermo-elastodynamics. The theory deals with the motion of an initially slender, anisotropic, heterogeneous and elastic beam of uniform cross-section. In the derivation, the customary Bernoulli–Euler hypothesis and its contradictions are eliminated, but the effects of transverse shear, transverse normal strains and rotatory inertia are included. The theory consists of the macroscopic beam equations of motion, the initial and natural boundary conditions, the strain–displacement relations and the constitutive equations.

The theory is established in a consistent manner by means of a series expansion method and a generalized variational theorem. It follows from the foregoing analysis that the use of series expansion for kinematic variables is indeed comprehensive and tractable. The variational theorem serves as an averaging procedure and it yields the equations of motion as well as the natural boundary conditions in a systematic manner. However, these equations can be constructed by the direct integration of the field equations as have been exhibited by Warner [30] and Antman and Warner [32] for beams and by the author [17] for plates and shells. Furthermore, the series expansion technique might be used for any other field quantities in lieu of kinematic variables as a starting point. This, of course, requires that one includes the usual compatibility conditions in the analysis. Moreover, the theory can similarly be formulated by the use of the direct method and the method of asymptotic expansion as was already noted.

In Section 9, it was shown that the linear version of the theory includes the familiar Bernoulli and Timoshenko beam theories as well as the Mindlin beam theory, as special cases. Also, the isothermal linear theory contains the theories derived by Warner [30] and more recently by Bleustein and Stanley [31], and it recovers the beam equations, up to order (1, 1), due to Medick [25, 26], Hertelendy [27] and Volterra [41, 42].

The theory is approached within a general framework. Consequently, obtaining a series of approximate results by simplification in the physical aspect and the kinematic description of the general theory as already pointed out, as well as extending the theory in some different directions is straightforward. First, two special cases of importance are mentioned: one is the counterpart of the Kármán plate equations in beams; that is to say, a nonlinear theory of beams derived by the use of the plane stress assumption and the Bernoulli–Euler hypothesis. The second case of interest is the one in which both extension and shear deformation are small compared to unity. In this instance, the products $e_{rk}e_{rl}$ and $e_{rk}\omega_{rl}$ are small in comparison to $\omega_{rk}\omega_{rl}$ and can therefore be omitted. This approximation then gives the following form for the strain tensor:

$$\gamma_{kl} = e_{kl}+\tfrac{1}{2}\omega_{rk}\omega_{rl}. \tag{10.1}$$

It should be noted that this partially geometrical nonlinearity leads to simpler equations of motion and stress boundary conditions:

$$\begin{aligned} (s_{kl}+\omega_{kr}s_{rl})_{,l}+\rho f_k &= \rho\ddot{u}_k \quad \text{in } \vartheta \\ n_l(s_{kl}+\omega_{kr}s_{rl}) &= t_k^* \quad \text{on } \mathscr{S}_\sigma. \end{aligned} \tag{10.2}$$

These equations are obtained through the Hamiltonian principle.

The theory presented here accommodates nonlinear torsional motions, in accordance with (a) the Saint-Venant theory of torsion of rods and (b) the Vlasov theory of thin-walled beams. If the loading is at the faces, and the body force and inertia terms are taken equal to zero, i.e. $Q_k^{(m,n)} = \ddot{U}_k^{(m,n)} = 0$ in (8.9) and the thermal terms are dropped, the nonlinear theory of the Saint-Venant torsion can readily be obtained by the use of the displacement field (cf. [35, 46]):

$$u_1 = x_2\,.\,u_1^{(0,1)}(x_3), \qquad u_2 = x_1\,.\,u_2^{(1,0)}(x_3), \qquad u_3 = \sum_{m+n=0}^{\infty} x_1^m x_2^n u_3^{(m,n)}(x_3) \tag{10.3a}$$

with

$$u_1^{(0,1)} = -u_2^{(1,0)} = -\omega\,.\,x_3, \qquad u_3^{(m,n)} = \omega\,.\,C_{mn}. \tag{10.3b}$$

Here, $\omega = \omega_{12,3}$ denotes the uniform rate of twist and C_{mn} is a constant. The usual warping function $\phi(x_\alpha)$ is expressed by

$$\phi(x_1, x_2) = \sum_{m+n=0}^{\infty} C_{mn} x_1^m x_2^n. \tag{10.3c}$$

In this connection, it is worthwhile to note that by the use of the same kinematic expressions, the thin plate theory and the Kármán theory for large deflection of plates have been established on the basis of linear and nonlinear elasticity theory, respectively, (see, e.g. [46]). Within the context of these approximations, the linear theory of higher order torsions has been recently examined in [31], as mentioned above. The Saint-Venant theory of torsion is well developed to cover the torsion of thin-walled beams. This is due to Vlasov [43]. In a recent paper, by the use of the kinematic expressions given in [43], Popelar [44] presented a partially nonlinear energy formulation. A modified derivation which eliminates the customary assumption of rigid cross-sections but also includes the shear effects can readily be obtained with the aid of the following expressions:

$$\begin{aligned} u_1 &= u_1^{(0,0)} + x_2 u_1^{(0,1)}, \qquad u_2 = u_2^{(0,0)} + x_1 u_2^{(1,0)}, \\ u_3 &= u_3^{(0,0)} \,.\, \phi(x_\alpha) + x_1 \,.\, u_3^{(1,0)} + x_2 \,.\, u_3^{(0,1)} \end{aligned} \tag{10.4}$$

and (10.3c) for the displacement components (cf. [44]).

Furthermore, the constitutive relation (7.1) can be used for a class of plasticity problems as noted in Section 7. For other inelastic materials, the constitutive equations can be constructed in a manner similar to the development given in Section 7. Moreover, the initial stress problem which is of special importance in the stability analysis of columns might be analyzed, if the line of attack presented is carried back to incremental field quantities [45, 47]. In like manner, the nonlinear theory of thin beams can similarly be formulated with the aid of a degenerate series (8.12), as remarked previously. Lastly, the extension of this theory to Cosserat media and to composites is also straightforward. This has been done for Cosserat plates and shells [17, 48, 49] and for composite beams [50], as a generalization of the case of Boley and Testa [51, 52].

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Абстракт—На основе трехмерной теории термо-зластодинамики определяются нелинейные основные уравнения, касающиеся движения балок, в виде состояния отношения. Используется общий метод обобщения, вместе с вариационном способом. Затем, постоянно устанавливается иерархия одномерных приближенных теорий. В анализе, обсуждаются подробно как геометрическия, так и физические нелинейности. Из общих результатов выводятся непосредственно классические теории напряжении и деформации в балках. Теория прцсиособицваем растяжение, изгиб и кручение высщего порядка для балок с постоянным поперечным сечением.